Deduce that the cheapest way of carrying the cargo on the required route is to choose $v = v^*$ *where*

$$v^* = \left(\frac{\alpha + \beta A}{\beta C}\right)^{1/2}.$$

What happens to v^* *as* α *increases?*

For some time tankers which were to be manned by United States crews were built with more powerful engines than those sailing with British crews. The United States crews had higher wages and consequently higher v^*.

The increasing power of the computer may eventually make the job of the modellers obsolete together with their water channels, water flumes, water tunnels, wind tunnels, shock tunnels, ship towing tanks, high-speed railways, annular water channels, whirling arms, tethered dynamic models, steering basins, ditching tanks, dropped bodies, fired and rocket driven models, spinning tunnels, surge tanks and tidal model basins, but for many years their mixture of science, art and a dash of black magic were essential to the progress of engineering.

There are various points that I have brushed aside in this discussion. The more mathematical ones like the question of how to find a complete set of dimensionless variables or how we know that each complete set will have the same number of variables are resolved by the methods of abstract linear algebra as taught in the first or second year of a university mathematics course†.

A more important problem is the choice of variables. How do we know that the key variables for a hovering helicopter are the power, the length of the rotor blades, the weight of the helicopter and the density of air? Clearly the colour of the pilot's eyes is unimportant, but what about air pressure and viscosity? In real life, dimensional analysis begins with a careful study of the equations involved and depends on long experience.

Consider the example of the simple pendulum with which we began. I took as key variables the length l of the pendulum the mass m of the attached weight and the acceleration g due to gravity. I ignored the amplitude (length) of the swing because 'as everyone knows' the period t is independent of the amplitude (for small swings). However, the independence is not an obvious fact and was the first great discovery of Galileo (made at the age of 19). Before closing my eyes to meditate on the pendulum I took a surreptitious peek at the real world‡.

In 1953, Einstein composed a preface to Galileo's *Dialogue Concerning the Two Chief World Systems*. The modern master of physics in a darkened room wrote:

†The complex of ideas is called the Buckingham π theorem. Like many such results it exists in many different forms. The classical exposition is in Birkhoff's *Hydrodynamics* but Chapter 1 of Logan's *Applied Mathematics* may be more accessible. The treatment of dimension I have given follows the traditional pattern in glossing over certain points. In [261] Barenblatt takes a more modern approach and shows that, if we think a little harder, we can understand a lot more.

‡We can get a little further using dimensional analysis as I shall show in Exercise 8.2.5.

There is no empirical method without speculative concepts and systems; and there is no speculative thinking whose concepts do not reveal, on closer investigation, the empirical material from which they stem.

6.2 A different age

How long would you expect a paper reporting a crucial experiment in physics to be and how would you expect it to be written? Here in its entirety is a paper entitled *Interference Fringes With Feeble Light* written by G. I. Taylor in 1909 (to be found in his collected works).

The phenomenon of ionisation by light and by Röntgen rays has led to a theory according to which energy is distributed unevenly over a wave-front. There are regions of maximum energy widely separated by large undisturbed areas. When the intensity of light is reduced these regions become more widely separated; but the amount of energy in any one of them does not change; that is they are indivisible units.

So far, all the evidence brought forward in support of the theory has been of an indirect nature; for all ordinary optical phenomena are average effects, and are therefore incapable of differentiating between the usual electromagnetic theory and the modification of it that we are considering. Sir J. J. Thomson, however, suggested that if the intensity of light in a diffraction pattern were so greatly reduced that only a few of these indivisible units of energy should occur on a Huygens zone at once, the ordinary phenomena of diffraction would be modified. Photographs were taken of the shadow of a needle, the source of light being a narrow slit placed in front of a gas flame. The intensity of light was reduced by means of smoked glass screens.

Before making any exposures it was necessary to find out what proportion of the light was cut off by these screens. A plate was exposed to direct gas light for a certain time. The gas flame was then shielded by the various screens that were to be used, and other plates of the same kind were exposed till they came out as black as the first plate on being completely developed. The times of exposure necessary to produce this result were taken as inversely proportional to the intensities. Experiments made to test the truth of this assumption showed it to be true if the light was not too feeble.

Five diffraction photographs were then taken, the first with direct light and the others with the various screens inserted between the gas flame and the slit. The time of exposure for the first photograph was obtained by trial, a certain standard of blackness being attained by the plate when fully developed. The remaining times of the exposure were taken from the first in the inverse ratio of the intensities. The longest time was 2000 hours or about 3 months. In no case was there any diminution in the sharpness of the pattern although the plates did not all reach the standard blackness of the first photograph.

In order to get some idea of the energy of the light falling on the plates in these experiments, a plate of the same kind was exposed at a distance of two metres from a standard candle till complete development brought it up to the standard of blackness. Ten seconds sufficed for this. A simple calculation will show that the amount of energy falling on the plate during the longest exposure was the same as that due to a standard candle burning at a distance slightly exceeding a mile. Taking the value given by Drude for the energy in the visible part of the spectrum of a standard candle, the amount of energy falling on 1 cm^2 of the plate is 5×10^{-6} ergs/sec and the amount of energy per cm^3 of this radiation is 1.6×10^{-16} ergs.

According to Sir J. J. Thomson, this value sets an upper limit on the amount of energy contained in one of the indivisible units above.

In 1905, Einstein, then 'Technical Expert, Third Class' at the Bern Patent Office, wrote papers announcing his theory of special relativity, a proof via Brownian motion of the physical existence of molecules and an explanation of the photo-electric effect in terms of a quantum theory of light. The scientific community was readily convinced of the truth of the first two theories but remained sceptical of the third for 20 years.

Einstein proposed that light was made up of individual particles or quanta which we now call photons. This seemed totally contrary, not simply to the immensely successful Maxwell theory which treated light as the wavelike propagation of an electromagnetic disturbance, but to a 100 years of observation of the wave character of light. Particles travel in straight lines but waves spread out and the cancellation of troughs with crests and the reinforcement of crests by crests and troughs by troughs creates the 'interference patterns' which characterise wave motion. If we shine a thin beam of light at a needle we see, not the sharp shadow that we expect a stream of particles to produce, but the interference patterns typical of wave propagation.

One way of reconciling Einstein's proposal with this observation was the suggestion that 'optical phenomena are average effects' and that large numbers of jostling photons somehow produce the observed wavelike phenomena. (After all, sound and water waves are produced by large numbers of particles.) When G. I. Taylor, fresh from Cambridge undergraduate study of mathematics and physics, sought a research project J. J. Thomson suggested that he investigate whether, when the intensity of the light beam is so reduced that the putative photons are widely separated, the interference patterns vanish. (After all, sound will

not travel through a near vacuum, and a few molecules of water will not form a wave.) Taylor set up the experiment at home with equipment costing under a pound and showed, as the paper above reports, that incredibly weak light beams produce the same interference patterns as strong ones. In later life he claimed that the reason he chose the project was that it left him free to go on a month's sailing cruise while the experiment was running.

The experiment seems to rule out the existence of photons conclusively, but the truth is much stranger. Experiments in the 1920s showed what had to be interpreted as collisions between electrons and photons and we are now convinced that light is made up of photons. In G. I. Taylor's experiment, the photons would have an energy of about 3×10^{-12} ergs and so, looking at the last but one sentence of Taylor's paper, we see that his dimmest beam of light contained roughly one photon per 10 000 cubic centimetres. Somehow, individual photons retain wavelike characteristics†. Thus what J. J. Thompson and G. I. Taylor presumably considered as a journeyman experiment confirming, once again, a classical theory and squashing wild speculations which had somehow got into circulation is now considered a central experiment in the new theory which replaced it.

†If you wish to learn how modern physics deals with the problem, you could start with Feynman's little book *QED, The Strange Theory of Light and Matter.*

After writing his first paper in the manner just described, Taylor returned to his main interests which lay not in the 'high physics' of the search for fundamental laws but in the equally hard task of using them to understand the world around us. 'While still at school I came across Lamb's *Hydrodynamics* in my uncle['s] ... library and though I could not understand it I was fascinated by its subject and hoped I would be able one day to use it in understanding the mechanics of sailing boats, a subject in which I was already much interested from the practical point of view.'‡ Initially, he worked on meteorology (including a six month expedition to study icebergs following the *Titanic* disaster). At the beginning of the First World War he offered his services to the army to set up a weather forecasting unit in the field. The officer to whom he made this proposal

‡The quotation comes from Batchelor's obituary of G. I. Taylor which I have used extensively in this chapter.

> did not seem to doubt that I could tell what the weather was going to be — as he might very well have done — but thought the knowledge would be of no value in the field. 'Soldiers don't go into battle under umbrellas, they go whether it is raining or not.'

Instead, he joined the Royal Aircraft Factory at Farnborough. Here, among other things, he was involved in a project (insisted on by higher authority) to develop a dart to be dropped from an aeroplane on enemy troops below. After a trial drop of a bundle of darts, he and a colleague